Mid-Semester Exam - Measure Theoretic Probability
Total marks: 35 - Time: 2h30m
Answer any five questions and each question is worth 7 marks

1. Let $(X, \mathcal{B}, \mu)$ be a measure space.

   (i) if A and B are measurable sets such that $A \subset B$, then show that $\mu(A) \leq \mu(B)$.

   (ii) if (E_i) is a countable collection of measurable sets, then show that there is a disjoint collection (F_i) of measurable sets such that $F_i \subset E_i$, $\cup E_i = \cup F_i$ and $\mu(\cup E_i) \leq \sum \mu(E_i)$.

2. Let $E \subset \mathbb{R}$ such that $m^*(E) < \infty$ where m^* is the Lebesgue outer measure. Then show that E is Lebesgue measurable if and only if to each $\epsilon > 0$, there exists $U \subset \mathbb{R}$ such that U is a finite union of open intervals and $m^*(U \Delta E) < \epsilon$ where $U \Delta E = U \setminus E \cup E \setminus U$.

3. Let $(X, \mathcal{B})$ be a measurable space. Let f and g be two real-valued measurable functions. Then show that $f+g$, $f-g$ and fg are measurable functions.

4. Let (f_n) be a sequence of non-negative measurable functions on a measure space $(X, \mathcal{B}, \mu)$.

   (i) Show that $\int (\liminf f_n) d\mu \leq \liminf \int f_n d\mu$.

   (ii) In addition if F is an integrable function such that $f_n \leq F$ a.e on X, then prove that $\int (\liminf f_n) d\mu \leq \liminf \int f_n d\mu \leq \limsup \int f_n d\mu \leq \int (\limsup f_n) d\mu$.

   (iii) Give a counter-example to show that $\limsup \int f_n d\mu \leq \int (\limsup f_n) d\mu$ is not always true for sequences of non-negative functions.

5. (i)Let $(X, \mathcal{B}, \mu)$ be a finite measure space. Let (f_n) be a sequence of bounded real-valued measurable functions and (f_n) converges uniformly to a function f on X. Then show that f_n is uniformly bounded and $\lim \int f_n = \int f < \infty$.

   (ii) Let f be a Riemann-integrable function on a bounded interval $[a, b]$. Then show that f is measurable.

6. (i) Let Z be a non-empty set and $\mathcal{M}$ be a non-empty collection of subsets of Z. Then show that there is a smallest monotone class containing $\mathcal{M}$.

   (ii) Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \lambda)$ be two complete measure spaces. Show that there is a smallest monotone class $\mathcal{M}$ containing all elementary sets in $X \times Y$ and $\mathcal{M}$ is a σ-algebra.

7. Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \lambda)$ be two complete σ-finite measure spaces. Let h and g be real-valued integrable functions on X and Y respectively. Let $f(x, y) = h(x)g(y)$ for all $(x, y) \in X \times Y$. Then show that f is integrable on $X \times Y$ and $\int f d(\mu \times \lambda) = (\int h d\mu)(\int g d\lambda)$.